MIND ACTION SERIES

MATHEMATICS WORKSHOP

EUCLIDEAN GEOMETRY
TEXTBOOK GRADE 11
(Chapter 8)

Presented by: Jurg Basson

Attending this Workshop = 10 SACE Points
CHAPTER 8  EUCLIDEAN GEOMETRY

BASIC CIRCLE TERMINOLOGY

Radius: A line from the centre to any point on the circumference of the circle.
Chord: A line with end-points on the circumference.
Diameter: A chord passing through the centre of the circle. It is double the length of the radius.
Tangent: A line touching the circle at only one point.
Secant: A line passing through two points on the circle.

THEOREMS INVOLVING THE CENTRE OF A CIRCLE

THEOREM 1A
The line drawn from the centre of a circle perpendicular to a chord bisects the chord. (line from centre ⊥ to chord)

If OM ⊥ AB then AM = MB

Proof
Join OA and OB.
In ΔOAM and ΔOBM:
(a) OA = OB radii
(b) M1 = M2 = 90° given
(c) OM = OM common
∴ ΔOAM ≡ ΔOBM RHS
∴ AM = MB

THEOREM 1A (Converse)
The line segment joining the centre of a circle to the midpoint of a chord is perpendicular to the chord. (line from centre to midpt of chord)

If AM = MB then OM ⊥ AB
**Definition**

The perpendicular bisector of a line is a line that bisects the given line at right angles. In the diagram, OM is the perpendicular bisector of AB.

**THEOREM 1B**

The perpendicular bisector of a chord passes through the centre of the circle. (perp bisector of chord)

**EXAMPLE 1**

O is the centre. AB = 8 cm, OF = 3 cm, OE = 4 cm, AF = FB and CD ⊥ OE. Calculate the length of chord CD.

**Solution**

\[ AF = 4 \text{ cm} \quad \text{AB} = 8 \text{ cm} \quad \text{and} \quad AF = FB \]
\[ \therefore \hat{F} = 90^\circ \quad \text{line from centre to midpt of chord} \]
\[ OA^2 = (3)^2 + (4)^2 \quad \text{Pythagoras} \]
\[ \therefore OA = 5 \text{ cm} \]
\[ \therefore OD = 5 \text{ cm} \quad \text{equal radii} \]
\[ ED^2 = (5)^2 - (4)^2 \quad \text{Pythagoras; } \hat{E} = 90^\circ \]
\[ \therefore ED = 3 \text{ cm} \]
But DE = CE \quad \text{line from centre } \perp \text{ to chord}
\[ \therefore CD = 6 \text{ cm} \]

**EXERCISE 1**

In all questions, O is the centre.

(a) Calculate the length of AC.

(b) Calculate the length of DE.

(c) Calculate the length of the radius of the circle and hence PQ.

(d) Determine the radius OB in terms of x and hence the length of OB.
(e) The radius of the semi-circle centre O is 5 cm. A square is fitted into the semi-circle as shown in the diagram. Calculate the area of the square. Let the length of the square equal \( x \).

![Diagram of semi-circle with square](image)

(f) AB is a chord of the circle. AC = CB, and the length of OA is 5 units. AC = 3 units and OC = 4 units. Show that OC \( \perp \) AB and explain why OC passes through the centre O.

![Diagram of circle with chord and radius](image)

**Subtended Angles**

Arc APB or chord AB subtends \( \hat{A} \hat{C}B \) at the circumference of the circle and \( \hat{A} \hat{O}B \) at the centre.

**THEOREM 2**

The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference (on the same side of the chord as the centre). \( (\angle \text{ at centre} = 2 \times \angle \text{ at circ}) \)

![Diagram of angles](image)

**Proof**

Consider Diagram 1 and Diagram 2. Join CO and produce.

\[ \hat{O}_1 = \hat{C}_1 + \hat{A} \] ex t \( \angle \) of \( \triangle AOC \)

Now OA = OC radii equal

\[ \therefore \hat{C}_1 = \hat{A} \] \( \angle \) s opp = sides

\[ \therefore \hat{O}_1 = 2\hat{C}_1 \]

Similarly, in \( \triangle OCB \), \( \hat{O}_2 = 2\hat{C}_2 \)

\[ \therefore \hat{O}_1 + \hat{O}_2 = 2\hat{C}_1 + 2\hat{C}_2 \]

\[ \therefore \hat{O}_1 + \hat{O}_2 = 2(\hat{C}_1 + \hat{C}_2) \]

\[ \therefore \hat{A} \hat{O}B = 2\hat{A} \hat{C}B \]

Consider Diagram 3. Join CO and produce.

\[ \hat{O}_1 = \hat{C}_1 + \hat{A} \] ex t \( \angle \) of \( \triangle AOC \)

Now OA = OC radii equal

\[ \therefore \hat{C}_1 = \hat{A} \] \( \angle \) s opp = sides

\[ \therefore \hat{O}_1 = 2\hat{C}_1 \]
Similarly, in $\triangle OCB$, $\hat{O}_2 = 2\hat{C}_2$

$\therefore \hat{O}_2 - \hat{O}_1 = 2\hat{C}_2 - 2\hat{C}_1$

$\therefore \hat{O}_2 - \hat{O}_1 = 2(\hat{C}_2 - \hat{C}_1)$

$\therefore AO \hat{B} = 2A\hat{C} \hat{B}$

**THEOREM 3**

The angle subtended at the circle by a diameter is a right angle.

We say that the angle in a semi-circle is $90^\circ$. (*\angle \text{ in a semi-circle})*

If $AB$ is a diameter then $\triangle A\hat{C}B = 90^\circ$

**THEOREM 3 (Converse)**

If the angle subtended by a chord at a point on the circle is $90^\circ$, then the chord is a diameter. (*Chord subtends $90^\circ$*)

If $\triangle A\hat{C}B = 90^\circ$ then $AB$ is a diameter

**EXAMPLE 2**

$O$ is the centre of each circle. Calculate the sizes of the angles marked with small letters.

**Solutions**

\[ a = 65^\circ \quad \angle \text{ at centre } = 2 \times \angle \text{ at circ} \]

\[ b = 25^\circ \quad \angle \text{ at centre } = 2 \times \angle \text{ at circ} \]

\[ \hat{O}_1 = 260^\circ \quad \angle \text{ at centre } = 2 \times \angle \text{ at circ} \]

\[ c = 360^\circ - 260^\circ \quad \angle \text{s round a point} \]

$\therefore c = 100^\circ$

\[ \hat{C} = 90^\circ \quad \angle \text{ in a semi-circle} \]

\[ d = 30^\circ \quad \text{int } \angle \text{s of } \triangle \]
EXERCISE 2

Calculate the value of the unknown variables. O is the centre in each case.

(a) \( \angle BAO = 60° \)

(b) \( \angle BAO = 24° \)

(c) \( \angle BAO = 10° \)

(d) \( \angle ABC = 36° \)

(e) \( \angle ABC = 120° \)

(f) \( \angle ABC = 230° \)

(g) \( \angle BAC = 150° \)

(h) \( \angle BAC = 95° \)

(i) \( \angle BAC = 70° \)

(j) \( \angle BAC = 20° \)

(k) \( \angle BAC = 110° \)

(l) \( \angle BAC = 55° \)

(m) \( \angle BAC = 52° \)

(n) \( \angle BAC = 104° \)

(o) \( \angle BAC = 60° \), \( \angle BAC = 22° \)
EXERCISE 3  (Challenges)

O is the centre of each circle.

(a) \[ \hat{O}_1 = 2\hat{C} \]
(b) \[ 2\hat{B}_1 + \hat{O}_1 = 180^\circ \]
(c) \[ \hat{O}_1 = 360^\circ - 2\hat{B} \]

Prove:
(1) \[ \hat{O}_1 = 2\hat{C} \]
(2) \[ \hat{B} + \hat{D} = 90^\circ \]

THEOREMS INVOLVING CYCLIC QUADRILATERALS

Cyclic quadrilaterals

A quadrilateral whose vertices lie on the circumference of a circle is referred to as a cyclic quadrilateral. ABCD is a cyclic quadrilateral because A, B, C and D are concyclic (lie on the circumference).

THEOREM 4

Angles subtended by a chord (or an arc) of the circle, on the same side of the chord (or the arc), are equal. \( \text{(\textit{\$s \text{ in the same seg}})} \)

In the diagram, \( \hat{A}_1 = \hat{B}_1 \), \( \hat{A}_2 = \hat{D}_2 \), \( \hat{C}_1 = \hat{D}_1 \) and \( \hat{B}_2 = \hat{C}_2 \)

THEOREM 4 (Converse)

If a line segment joining two points subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic (lie on the circumference). ABCD will be a cyclic quadrilateral. \( \text{(\textit{line subt = \$s})} \)
Corollaries

(a) Equal chords subtend equal angles at the circumference.
   (equal chords ; equal \( \angle \)s)

(b) Equal chords subtend equal angles at the centre.
   (equal chords ; equal \( \angle \)s)

(c) Equal chords of equal circles subtend equal angles
    at the circumference.
   (equal circles ; equal chords ; equal \( \angle \)s)

\[ \hat{A} = \hat{D} \quad \text{and} \quad \hat{G} = \hat{H} \]

EXAMPLE 3
Calculate the value of the unknown angles.

Solutions

\[ x = 60^\circ \quad \angle \text{ in the same segment} \]
\[ y = 130^\circ \quad \text{equal chords, equal angles} \]
\[ z = 60^\circ \quad \angle \text{ opp } = \text{ sides} \]

EXERCISE 4
Calculate the value of the unknown angles. O is the centre.

(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d) \hspace{1cm} (e) \hspace{1cm} (f)
The two circles have diameters that are equal. O is the centre of circle ABE.

**EXERCISE 5**  (Challenges)

(a) Prove:
(1) \( \hat{E} = \hat{B} \)
(2) \( \hat{BDF} = \hat{A} + \hat{E} \)

(b) Prove:
\( \hat{B} = 90^\circ - \hat{O} \)
(O is the centre)

(c) Prove:
\( \hat{A} = 2\hat{B} - 90^\circ \)
(O is the centre)

In the diagram, chord EF = chord GH.
A, B, H, G, F and E are concyclic.
Prove that ABCD is a cyclic quadrilateral.
In the diagram, O and P are the centres of two circles intersecting at C and EC = ED. AB and BC are chords of the larger circle. CD is a chord of the smaller circle. Prove that ABEP is a cyclic quadrilateral.

**THEOREM 5**

The opposite angles of a cyclic quadrilateral are supplementary (add up to 180°).

*(opp ∠'s of cyclic quad)*

If AB is a cyclic quadrilateral then the opposite angles are supplementary \( \hat{B} + \hat{D} = 180° \) and \( \hat{A} + \hat{C} = 180° \).

**Proof**

In cyclic quadrilateral ABCD, join AO and OC.

\[ \hat{O}_1 = 2\hat{D} \quad \hat{\angle} \text{ at centre} = 2\times \hat{\angle} \text{ at circ} \]

\[ \hat{O}_2 = 2\hat{B} \quad \hat{\angle} \text{ at centre} = 2\times \hat{\angle} \text{ at circ} \]

\[ \therefore \hat{O}_1 + \hat{O}_2 = 2\hat{D} + 2\hat{B} \]

and \( \hat{O}_1 + \hat{O}_2 = 360° \quad \hat{\angle} \text{ s round a point} \)

\[ \therefore 360° = 2\hat{D} + 2\hat{B} \]

\[ \therefore 180° = \hat{D} + \hat{B} \]

Similarly, by joining BO and DO, it can be proven that \( \hat{A} + \hat{C} = 180° \).

**THEOREM 5** *(Converse)*

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral. *(opp ∠'s quad supp)*

If the opposite angles of quadrilateral ABCD are supplementary then ABCD is a cyclic quadrilateral.

**THEOREM 6**

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. *(ext ∠ of cyclic quad)*

If ABCD is a cyclic quadrilateral then \( \hat{C}_1 = \hat{A} \).
THEOREM 6 (Converse)

If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is a cyclic quadrilateral. \((\text{ext } \angle = \text{ int opp } \angle)\)

If \(\hat{C}_1 = \hat{A}\) then \(ABCD\) is a cyclic quadrilateral

EXAMPLE 4

ABCD and BCDE are cyclic quadrilaterals. Calculate the value of the unknown angles.

Solutions

\[
\begin{align*}
a &= 80^\circ & \text{opp } \angle \text{ s of cyclic quad } ABCD \\
b &= 70^\circ & \text{ext } \angle \text{ of cyclic quad BCDE} \\
c &= 80^\circ & \angle \text{ s in the same segment} \\
& & \text{(or opp } \angle \text{s of cyclic quad BCDE)} \\
d &= 30^\circ & \angle \text{s in the same segment} \\
e &= 70^\circ & \text{int } \angle \text{s of } \Delta
\end{align*}
\]

EXERCISE 6

Calculate the value of the unknown angles. O is the centre.

(a) \(83^\circ\) \(x\) \(y\) \(105^\circ\) \(x\) \(y\)
(b) \(82^\circ\) \(z\) \(y\) \(40^\circ\) \(62^\circ\)
(c) \(110^\circ\) \(y\) \(x\) \(m\) \(150^\circ\)
(d) \(80^\circ\) \(x\) \(y\)
(e) \(100^\circ\) \(z\) \(y\) \(2x\) \(y\)
(f) \(80^\circ\) \(x\) \(y\) \(z\) \(x\)
(g) \(84^\circ\) \(x\) \(y\) \(z\) \(x\)
(h) \(150^\circ\) \(y\) \(x\) \(z\) \(y\)
(i) \(80^\circ\) \(x\) \(y\) \(z\) \(x\)
EXERCISE 7  (Challenges)

O is the centre of each circle.

Prove:

(1) \( \hat{A}_1 + \hat{D} = 180^\circ \)

(2) \( \hat{O}_1 = 360^\circ - 2\hat{A}_1 \)

Prove:

(1) \( \hat{O}_1 = 2\hat{Y}_1 \)

(2) \( \hat{Y}_2 = \hat{K} \)

Prove:

(1) \( \hat{O}_1 = 180^\circ - 2\hat{B}_1 \)

(2) \( \hat{O}_1 = 360^\circ - 2\hat{D} \)

**Important Summary (strategies to prove that a quadrilateral is cyclic)**

ABCD is a quadrilateral. One of the following three strategies can be used to show that ABCD a cyclic quadrilateral.

**Strategy 1:** Show that at least one pair of opposite angles are supplementary

\[
\hat{A} + \hat{C} = y + (180^\circ - y) = 180^\circ \\
\text{or} \\
\hat{B} + \hat{D} = x + (180^\circ - x) = 180^\circ
\]
Strategy 2: Show that an exterior angle is equal to the interior opposite angle.

For example, if $\hat{C}_1 = \hat{A}$, then $ABCD$ will be a cyclic quadrilateral.

Strategy 3: Show that a line segment subtends equal angles on the same side of that line segment.

For example, since it is given that line segment $AB$ subtends equal angles ($\hat{C}_1 = \hat{D}_1$), we can conclude that $ABCD$ is a cyclic quadrilateral.

EXERCISE 8

(a) Show that $LMRQ$ is a cyclic quadrilateral if $PQ = PR$ and $LM \parallel QR$.

(b) Write down, with reasons, two cyclic quadrilaterals in the figure.

(c) In the diagram, figure $ABCD$ is a parallelogram with $\hat{B} = 80^\circ$. $ABFE$ is a cyclic quadrilateral. Show that $DEFC$ is a cyclic quadrilateral.

(d) A circle through $C$ and $D$ cuts parallelogram $CDFG$ at $H$ and $I$. Prove that $GFIH$ is cyclic quadrilateral.
THEOREMS INVOLVING TANGENTS TO A CIRCLE

AXIOM 7

A tangent to a circle is perpendicular to the radius at the point of contact. (tan ⊥ radius)

If ABC is a tangent to the circle at B

AXIOM 7 (Converse)

If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then the line is a tangent to the circle. (line ⊥ radius)

THEOREM 8

Two tangents drawn to a circle from the same point outside the circle are equal in length. (tangents from same pt)

THEOREM 9

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment. (∠ between tangent and chord) or (tan-chord theorem)

If CDE is a tangent to the circle at D

Proof

Case 1 (Acute angles)

We will prove that CBD = B̂ED

Draw diameter BOF and join EF

Æ1 + Õ2 = 90° tangent ⊥ radius

Æ̂1 + Õ̂2 = 90° ∠ in semi-circle

But Õ̂1 = Õ̂

∴ Õ̂2 = Õ̂

∴ CBD = B̂ED

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Case 2  (Obtuse angles)
We will prove that \( \hat{\text{A}\hat{B}E} = \hat{\text{B}\hat{D}E} \)

Draw diameter BOF and join FD
\( \hat{\text{B}_1} = 90^\circ \)  \( \text{tangent} \perp \text{radius} \)
\( \hat{\text{D}_1} = 90^\circ \)  \( \angle \text{in semi-circle} \)
\( \therefore \hat{\text{B}_1} = \hat{\text{D}_1} \)
But \( \hat{\text{B}_2} = \hat{\text{D}_2} \)  \( \angle \text{s in the same segment} \)
\( \therefore \hat{\text{B}_1} + \hat{\text{B}_2} = \hat{\text{D}_1} + \hat{\text{D}_2} \)
\( \therefore \hat{\text{A}\hat{B}E} = \hat{\text{B}\hat{D}E} \)

**THEOREM 9** (Converse)

If a line is drawn through the endpoint of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle. (\( \angle \text{between line and chord} \) or (converse of tan-chord theorem)

**EXAMPLE 5**

Calculate the value of the unknown angles. ABC is a tangent to each circle.

(a)

\[ a = 55^\circ \]  \( \text{tan} \perp \text{radius} \)
\[ b = 90^\circ \]  \( \angle \text{in semi-circle} \)
\[ \hat{\text{F}_1} + c + 65^\circ = 180^\circ \]  int \( \angle \text{s of } \Delta \)
\[ \text{BC} = \text{FC} \]  \( \text{tangents from same pt} \)
\[ \hat{\text{F}_1} = \hat{\text{C}} \]  \( \angle \text{s opp} = \text{sides} \)
\[ \therefore 2c + 66^\circ = 180^\circ \]
\[ \therefore c = 57^\circ \]

(b)

\[ x = 72^\circ \]  \( \text{tan-chord theorem} \)
\[ y = 65^\circ \]  \( \text{tan-chord theorem} \)
**Solutions**

\[ x = 70^\circ \]

\[ \hat{B}_1 = 40^\circ \]

\[ y = \hat{B}_1 + x \]

\[ y = 40^\circ + 70^\circ \]

\[ y = 110^\circ \]

**EXERCISE 9**

Calculate the value of the unknown angles. O is the centre and ABT is a tangent.

(a) \( \angle ABT \)

(b) \( \angle OBT \)

(c) \( \angle ABT \)

(d) \( \angle CDT \)

(e) \( \angle DAB \)

(f) \( \angle DBA \)

(g) \( \angle EPQ \)

(h) \( \angle QRS \)

(i) \( \angle ECD \)

(j) \( \angle ABC \)

(k) \( \angle OAD \)
EXERCISE 10

(a) Prove that PT is a tangent to the circle if $\hat{P}_4 = 60^\circ$, $B\hat{C}P = 120^\circ$ and $\hat{C}_1 = \hat{C}_2$

(b) Prove that SRT is a tangent to the circle through PQR.

(c) In the diagram, S is the centre of the circle and $SQ \perp PR$. TP = 12, PR = TU = 8 and SQ = 3.
Prove that TP is a tangent to the circle.

(d) O is the centre of the circle through A, B, C and T. DTE is a tangent at T and chord BC = chord CT

(1) Why is $\hat{A}_1 = \hat{A}_2$?

(2) Prove that $\hat{O}_1 = 2\hat{T}_3$

(3) Prove that $\hat{T}_2 + \hat{B}_1 + \hat{C}_1 = 180^\circ$
(e) In the diagram, DE is a tangent to the circle ACD at D. BC||DE. Prove that CD is a tangent to the circle passing through A, B and C.

**EXERCISE 11   (NUMERICAL GEOMETRY PROBLEMS)**

(a) In the diagram below, AOB is a diameter of the circle centre O.

OD||BC. The length of the radius is 10 units.

(1) What is the size of \( \hat{C} \)? State a reason.
(2) What is the size of \( \hat{E_1} \)? State a reason.
(3) Why is \( AE = EC \)? State a reason.
(4) If \( AC = 16 \) units, calculate the length of ED.

(b) In the figure below, RDS is a tangent to the circle centre O at D.

\( BC = DC \) and \( \hat{CDS} = 40^\circ \)

(1) What is the size of \( \hat{B_1} \). State a reason.
(2) What is the size of \( \hat{D_2} \). State a reason.
(3) What is the size of \( \hat{C} \). State a reason.
(4) Calculate the size of \( \hat{O_2} \) State a reason.
(5) Calculate the size of \( \hat{O_1} \). State a reason.
(6) Calculate the size of \( \hat{D_3} \) State reasons.
(7) Calculate the size of \( \hat{A} \). State a reason.

(c) In the diagram, O is the centre of the circle passing through A, B, C and D. AB||CD and \( \hat{B} = 20^\circ \)

(1) Calculate the size of \( \hat{C_1} \)? State a reason.
(2) Calculate the size of \( \hat{O_1} \)? State a reason.
(3) Calculate the size of \( \hat{D} \)? State a reason.
(4) Calculate the size of \( \hat{E_1} \)? State a reason.
(5) Why is AOEC a cyclic quadrilateral?
(d) AB and CD are two chords of the circle centre O. OE \perp CD, AF = FB. OE = 4 cm, OF = 3 cm and AB = 8 cm. Calculate the length of CD.

(e) In the circle centre O, BO \perp OD, AE = EC and A\hat{O}D = 116^\circ. Calculate the size of:
   (1) \hat{C}
   (2) A\hat{B}C

(f) O is the centre of the circle. AB is the diameter. A\hat{O}C = 104^\circ and D\hat{A}B = 32^\circ. Calculate the size of:
   (1) \hat{D}
   (2) \hat{C}_3
   (3) A_1
   (4) \hat{C}_2

(g) O is the centre of the circle. STU is a tangent at T. Chord BC = chord CT, A\hat{T}C = 105^\circ and C\hat{T}U = 40^\circ. Calculate the size of:
   (1) A_2
   (2) A_1
   (3) B_1 + B_2
   (4) \hat{C}_2
(h) PQR is a tangent at Q.
ST \parallel QW.
WQR = 30° and TSW = 70°.
Calculate the size of the following angles with reasons:
(1) \hat{V}
(2) \hat{Q}_1
(3) \hat{T}_1
(4) \hat{W}_2

(i) O is the centre of circle ABC.
AC is produced to D so that CD = CB.
Prove that \hat{O}_1 = 128° if \hat{D} = 32°.

(j) QOS is a diameter of the circle centre O.
QR = RT and \hat{T} = 35°.
Calculate, with reasons, the size of:
(1) \hat{R}_2
(2) \hat{S}_2
(3) \hat{Q}_2
(4) \hat{Q}_1
(5) \hat{P}_1

(k) In the diagram, O is the centre of the circle and AO \parallel BC.
If \hat{A}_1 = 130°, calculate the size of \hat{O}_1.

(l) In the diagram below, JM \parallel KL and the centre of the circle is O.
\hat{O}_1 = 120°.
(1) Calculate the size of \hat{Y}_1.
(2) Prove that XYLK is a cyclic quadrilateral.
(GEOMETRY PROBLEMS INVOLVING VARIABLES)

(m) In the diagram, ABCD is a cyclic quadrilateral with AB parallel to chord ED. \( \hat{A}_1 = x \) and AB bisects \( \hat{NAG} \). TAN is a tangent to the circle at A.

1. Write down, with reasons, seven angles equal to \( x \).
2. Why is AB a tangent to the circle passing through A, G and E?

(n) In the diagram, chord PR is parallel to chord TS. PR = PS and PT = TS. ARB is a tangent to the circle at R and RST is a straight line. \( \hat{R}_3 = x \) and \( \hat{R}_1 = y \).

1. Write down, with reasons, three other angles equal to \( x \) and three other equal to \( y \).
2. Express \( \hat{T} \) in terms of \( y \).
3. Express \( \hat{T} \) in terms of \( x \).

(o) In the diagram, DE is a tangent to circle ABCD at D. AD \( \parallel \) BC and \( \hat{B}_3 = \hat{D}_3 \).

1. Why is \( \hat{D}_3 = x \)?
2. Why is DEFB a cyclic quadrilateral?
3. Why is \( \hat{B}_3 = x \)?
4. Why is \( \hat{D}_3 = x \)?
(Don’t assume that EB is a tangent!)
5. Why can we now conclude that EB is a tangent at B?

(p) In the diagram, O is the centre of the circle and E is the midpoint of chord AD. AO and EO are produced to cut the circle at C and B respectively. AD produced meets the tangent drawn to the circle at C in G. \( \hat{O}_3 = x \)

1. Why is EOOG is a cyclic quadrilateral?
2. Write down, with reasons, two other angles equal to \( x \).
3. Express \( \hat{A}_1 \) in terms of \( x \).
4. Express \( \hat{C}_1 \) in terms of \( x \).
5. Express \( \hat{C}_2 \) in terms of \( x \).
(q) O is the centre of the circle through
A, B, C and D. BC = CD and \( \hat{B}OD = 2x \).
Express the following in terms of \( x \).

1. \( \hat{B}_2 \)
2. \( \hat{BCD} \)
3. \( \hat{A} \)

(r) In the diagram, AB is a diameter of the circle.
EA is a tangent to the circle at A.
\( \hat{B}_1 = x \) and \( \hat{E} = y \).

1. Name one other angle equal to \( x \).
2. Show that \( \hat{A}_2 = y \).
3. Name one other angle equal to \( y \).

(s) In the diagram, AOB is the diameter of the
semi-circle AMNB. MO||NB and \( \hat{B}_1 = x \).

1. Express \( \hat{M}_2 \) and \( \hat{B}_2 \) in terms of \( x \).
2. Express \( \hat{O}_2 \) and \( \hat{N}_2 \) in terms of \( x \).
3. Express \( \hat{K}_1 \) in terms of \( x \).

*(4) If it is given that \( x = 30^\circ \), calculate
the sizes of the angles of \( \triangle MKN \).

*(5) Hence prove that MOBN is a rhombus.

**SOLVING GEOMETRICAL RIDERS**

**Some important concepts**

A **rider** is a geometrical problem requiring proofs of statements. The problem involves
the application of a combination of theorems and logical abstract reasoning. The
following statements of logic are extremely important for solving geometrical riders.

\[(a) \quad \text{If } a = b \quad \text{and } b = c \quad \text{then } a = c \]
\[(b) \quad \text{If } a + b = c \quad \text{and } a + d = c \quad \text{then } b = d \]
\[(c) \quad \text{If } a = b \quad \text{and } c = d \quad \text{then } a + c = b + d \]
\[(d) \quad \text{If } a + b = c + d \quad \text{and } b = c \quad \text{then } a = d \]
\[(e) \quad \text{If } a + b + c = d + e \quad \text{and } b = d \quad \text{then } a + c = e \]
\[(f) \quad \text{If } a + b + c = d \quad \text{and } a = b \quad \text{then } 2b + c = d \quad \text{or } 2a + c = d \]
EXAMPLE 1

CD is a tangent at C to the circle through A, B, C and E. BC = CE and chords AC and BE intersect at F. Prove that:
(a) \( \hat{C}_1 = \hat{A}_2 \)
(b) \( \hat{C}_4 = \hat{E}_2 \)

Solutions

(a) \( \hat{C}_1 = \hat{A}_1 \)  
   tan-chord theorem  
   \( \hat{A}_1 = \hat{A}_2 \)  
   equal chords ; equal angles  
   \( \therefore \hat{C}_1 = \hat{A}_2 \)

(b) \( \hat{C}_4 = \hat{A}_2 \)  
   tan-chord theorem  
   \( \hat{A}_2 = \hat{A}_1 \)  
   equal chords ; equal angles  
   \( \therefore \hat{C}_4 = \hat{A}_1 \)  
   \( \hat{A}_1 = \hat{E}_2 \)  
   \( \angle s \) in the same segment  
   \( \therefore \hat{C}_4 = \hat{E}_2 \)

Example 2

PR is a diameter of circle PRMS with centre Q. PS, SR and PM are chords. PM bisects RPS. Prove that:
(a) \( PS \parallel QM \)
(b) \( QM \perp SR \)
(c) \( QM \) bisects SR

Solutions

(a) \( \hat{P}_2 = \hat{M} \)  
   \( \angle s \) opp = radii  
   \( \hat{P}_2 = \hat{P}_1 \)  
   given  
   \( \therefore \hat{P}_1 = \hat{M} \)  
   \( \therefore PS \parallel QM \)  
   alt \( \angle s \) =

(b) \( \hat{S} = 90^\circ \)  
   \( \angle \) in semi-circle  
   \( \therefore \hat{E}_2 = 90^\circ \)  
   corr \( \angle s \) =  
   \( \therefore QM \perp SR \)

(c) \( RE = ES \)  
   line from centre \( \perp \) to chord  
   \( \therefore QM \) bisects SR
EXAMPLE 3

LOM is a diameter of circle LMT. The centre is O. TN is a tangent at T. LN $\perp$ NP. MT is a chord.
LT is a chord produced to P. Prove that:
(a) MNPT is a cyclic quadrilateral
(b) NP = NT

Solutions

(a) $\hat{N}_1 + \hat{N}_2 = 90^\circ$ given
$\hat{T}_3 = 90^\circ$ $\angle$ in semi-circle
$\therefore \hat{N}_1 + \hat{N}_2 = \hat{T}_3$
$\therefore$ MNPT is a cyclic quad ext $\angle$ of quad $=$ int opp $\angle$

(b) $\hat{T}_1 = \hat{T}_4$ vert opp $\angle$ s $=$
$\hat{T}_4 = \hat{M}_1$ tan-chord theorem
$\therefore \hat{T}_1 = \hat{M}_1$
$\hat{M}_1 = \hat{P}$ ext $\angle$ of cyclic quad
$\therefore \hat{T}_1 = \hat{P}$
$\therefore$ NP = NT sides opp $=$ $\angle$

EXAMPLE 4

AB, AC, DB, and DC are chords.
DE $\perp$ AC and DB $\perp$ FC.
Prove that:
(a) DEFC is a cyclic quadrilateral
(b) AB $\parallel$ EF

Solutions

(a) $\hat{E}_1 = 90^\circ$ given
$\hat{F}_2 = 90^\circ$ given
$\therefore$ DEFC is a cyclic quad DC subtends $=$ $\angle$s

(b) $\hat{E}_2 = \hat{D}_2$ $\angle$s in the same seg
$\hat{A} = \hat{D}_2$ $\angle$s in the same seg
$\therefore \hat{E}_2 = \hat{A}$
$\therefore$ AB $\parallel$ EF corr $\angle$s $=$
**EXAMPLE 5**

O is the centre of circle SAT which is inscribed in ΔPQR. PQ, QR and PR are tangents to the circle. Prove:
(a) PSOT is a cyclic quadrilateral  
(b) OS is a tangent to circle SPE

![Diagram of circle SAT with tangents PQ, QR, PR, and inscribed triangle ΔPQR]

**Solutions**

(a) \( \hat{T_1} + \hat{T_2} = 90^\circ \)  
\( \hat{S_1} + \hat{S_2} = 90^\circ \)  
\( \therefore \hat{T_1} + \hat{T_2} + \hat{S_1} + \hat{S_2} = 180^\circ \)  
\( \therefore \) PSOT is a cyclic quadrilateral  

(b) \( \hat{S_1} = \hat{T_1} \)  
\( \hat{P_2} = \hat{T_1} \)  
\( \therefore \hat{S_1} = \hat{P_2} \)  
\( \therefore \) OS is a tangent to circle SPE

**EXAMPLE 6**

A, D, C, B and P are concyclic. PC bisects \( \hat{D CB} \). Prove that PA bisects \( \hat{X AB} \).

**Solution**

\( \hat{A_1} = \hat{C_1} \)  
ext \( \angle \) of cyclic quad  
\( \hat{C_1} = \hat{C_2} \)  
given  
\( \therefore \hat{A_1} = \hat{C_2} \)  
\( \hat{C_2} = \hat{A_2} \)  
\( \angle \) in the same seg  
\( \therefore \hat{A_1} = \hat{A_2} \)
EXERCISE 12  (SOLVING RIDERS)

(a) ABC is a tangent to the circle BED. BE || CD.
Prove that \( \hat{D}_1 = \hat{C} \)

(b) In circle ABCDE, BC and AD are parallel chords. AB is produced to F.
(1) Name 2 cyclic quadrilaterals
(2) Prove that:
   (i) \( \hat{B}_1 = \hat{E} \)
   (ii) \( \hat{D}_1 = \hat{A} \)

(c) AC is a diameter of circle centre O. B is a point on the circle.
OP bisects AB.
Prove that OP || BC.

(d) PQB is a tangent to the circle QRS at Q. QS bisects BQR.
Prove that QS = RS

(e) ABCD is a cyclic quadrilateral. BA is produced to E. AD bisects \( \hat{E}AC \).
Prove that DC = DB.

(f) AQ is a tangent to the circle in Q and QP || AR. Prove that RA is a tangent to circle ABQ at A.
(g) In circle ABCD, AB = BC. Prove that AB is a tangent to circle AED in A.

(h) ALB is a tangent to circle LMNP. ALB || MP. Prove that:
   (1) LM = LP
   (2) LN bisects MNP
   (3) LM is a tangent to circle MNQ

(i) EC is a diameter of circle DEC. EC is produced to B. BD is a tangent at D. ED is produced to A and AB ⊥ BE.
   Prove that:
   (1) ABCD is a cyclic quadrilateral
   (2) \( \hat{A}_1 = \hat{E} \)
   (3) \( \Delta BDA \) is isosceles
   (4) \( \hat{C}_2 = \hat{C}_3 \)

(j) PA and PC are tangents to the circle at A and C. AD || PC, and PD cuts the circle B. CB is produced to meet AP at F. AB, AC and DC are drawn.
   Prove:
   (1) AC bisects \( \hat{P} \hat{A}D \)
   (2) \( \hat{B}_1 = \hat{B}_3 \)
   (3) \( \hat{A} \hat{P}C = \hat{A} \hat{B}D \)
(k) TA is a tangent to the circle PRT. 
M is the midpoint of chord PT. 
The centre of the circle is O. 
PR is produced to intersect TA at A 
and TA ⊥ PA. T and R are joined. 
OR and OT are radii. 
Prove that: 
(1) MTAR is a cyclic quadrilateral 
(2) PR = RT 
(3) TR bisects P_TA 
(4) \( T_2 = \frac{1}{2} \hat{O}_1 \)

**EXAMPLE 6  (MORE ADVANCED RIDERS)**

In the diagram, PA is a tangent 
to the circle at A. AC and AB are 
chords and AB is produced to K 
such that \( AK = AM \). K and M 
lie on AC and AB respectively. 
Chord CB is produced to P. 
Prove that KP bisects \( \hat{APC} \).

**Solution**

\( \hat{P}_1 + \hat{A}_1 = \hat{M}_1 \) 
ext \( \angle \) of \( \Delta AMP \)

But \( \hat{M}_1 = \hat{K}_1 \) 
\( \angle \) s opp = sides

\( \therefore \hat{P}_1 + \hat{A}_1 = \hat{K}_1 \)

But \( \hat{P}_2 + \hat{C} = \hat{K}_1 \) 
ext \( \angle \) of \( \Delta CKP \)

and \( \hat{A}_1 = \hat{C} \) 
tan-chord theorem

\( \therefore \hat{P}_1 = \hat{P}_2 \)

**EXERCISE 13**

(a) FC is a tangent to circle BCD and FG is a tangent to circle FEDC. The circles 
intersect at D and C. Chord BD is produced to F. DC is joined. Chord FE and 
chord CB produced meet at A. 
Prove that: 
(1) ABDE is a cyclic 
quadrilateral.
(2) \( \hat{D}_1 = \hat{C}_1 + \hat{C}_2 \)
(3) FG||AC
(b) LPN is a tangent to circle ADP. Circle BCP touches the larger circle internally at P. Chord AD cuts the smaller circle at B and C and BP and CP are joined. Prove that $\hat{P}_2 = \hat{P}_4$. 

\[ \text{CONSOLIDATION AND EXTENSION EXERCISE} \]

(a) In the diagram below, QP is a tangent to a circle with centre O. RS is a diameter of the circle and RQ is a straight line. T is a point on the circle. PS bisects $\hat{TQP}$ and $\hat{SPQ} = 22^\circ$. Calculate the following, giving reasons:

- (1) $\hat{P}_2$
- (2) $\hat{R}_2$
- (3) $\hat{P}_3 + \hat{P}_4$
- (4) $\hat{R}_1$
- (5) $\hat{O}_1$
- (6) $\hat{Q}_2$

(b) O is the centre of the circle ABED. The radius of the circle is 6,5 cm. AC is a tangent to the circle at A. BD is produced to C such that $AD = DC$. 

- (1) Prove that $AB = AC$.
- (2) If $BE = 5$ cm, calculate the length of AC.
(c) In the diagram below, \( \hat{A}_1 = \hat{C} \), \( \hat{B} = 3x + 10^\circ \) and \( \hat{D} = 2x + 20^\circ \).
Calculate, with reasons, the value of \( x \).

(d) In the figure, \( O \) is the centre of the circle. \( PRO \) is a straight line intersecting the circle at \( R \). \( OT \) is perpendicular to chord \( SQ \) at \( T \). \( SR \) is joined. Radius \( OQ \) and line \( PQ \) meet at \( Q \). \( PQ = 12 \) units, \( QS = 8 \) units, \( RP = 8 \) units and \( OT = 3 \) units.
(1) Prove that \( PQ \) is a tangent to the circle at \( Q \).
(2) Express \( \hat{P} \) in terms of \( x \) if \( \hat{Q}_4 = x \).

(e) \( AC \) is a diameter of the circle centre \( B \). \( FED \) is a tangent to the circle at \( E \) and \( BG \perp EC \). \( BG \) produced cuts \( FE \) produced at \( D \). \( DC \) is drawn.
Prove that:
(1) \( BG \parallel AE \)
(2) \( BCDE \) is a cyclic quadrilateral.
(3) \( DC \) is a tangent to circle \( EAC \).
(4) \( DC \) is a tangent to circle \( BCG \).

(f) Circles \( OAS \) and \( ABCS \) intersect at \( A \) and \( S \). \( SB \) bisects \( \hat{\triangle ABC} \).
Chord \( AT \) is produced to \( C \).
Prove that \( SA = SO \)

(g) \( P \) is a point on side \( AB \) of \( \triangle ABC \).
The circle through \( P, B \) and \( C \) cuts \( AC \) in \( Q \). \( QP \) produced cuts the larger circle at \( R \). Prove that \( \hat{P}_1 = \hat{A}_1 + \hat{B}_1 \).
(h) Two circles with centre O and P touch each other. The radii of the circles are 7 and 4 respectively. Calculate the length of AB if AB is a tangent to both circles. Round off your answer to one decimal place.

(i) BC is a diameter of circle ABC with centre O. OD ⊥ AC and AT ⊥ BOC. BT = x, TO = 5 and AD = √42.

1. Prove that OA = 7
2. Calculate the length of AB.

(j) PR is a common chord of circles PQSR and PXRY. PR bisects ∠XRY. T lies on chord PS such that ST = SR = SQ.

MPY touches the larger circle at P. Prove that:
1. ∠Q₁ = ∠Q₂
2. QP is a tangent to the smaller circle.

(k) In the figure, ABCD is a cyclic quadrilateral with AB = AD and DC = BC. DC and BC, both produced meet AB and AD, both produced, at E and F respectively.

AC produced meets FE at G with ∠G₁ = 90°. Prove that:
1. AC is a diameter of the circle.
2. DBEF is a cyclic quadrilateral.
3. BC bisects ∠DBG.